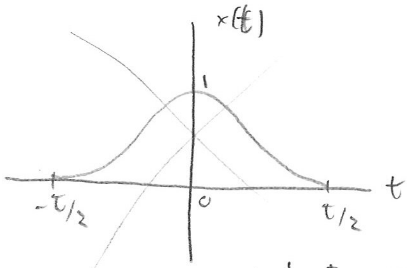
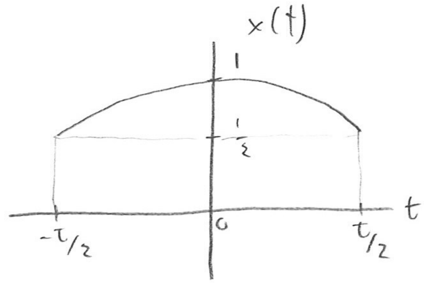
1.a



this is what I wanted it to be



1.b

$$x(t) = \frac{1}{2}\left(1+\cos\left(\frac{\pi t}{\tau}\right)\right)\Pi\left(\frac{t}{\tau}\right) = \left(\frac{1}{2} + \frac{1}{4}e^{2\pi j t/2\tau} + \frac{1}{4}e^{-2\pi j t/2\tau}\right)\Pi\left(\frac{t}{\tau}\right)$$

$$\Pi\left(\frac{t}{\tau}\right) \xrightarrow{\mathcal{F}} \tau\,\mathrm{sinc}(\tau f)$$

$$e^{\pm 2\pi j t/2\tau} \xrightarrow{\mathcal{F}} \tau\,\mathrm{sinc}\left(\tau\left(f \mp \frac{1}{2\tau}\right)\right)$$

$$\boxed{X(f) = \frac{\tau}{2}\mathrm{sinc}(\tau f) + \frac{\tau}{4}\mathrm{sinc}\left(\tau f - \frac{1}{2}\right) + \frac{\tau}{4}\mathrm{sinc}\left(\tau f + \frac{1}{2}\right)}$$

1.c

$$\int_{-\infty}^{\infty}|X(f)|^2 df = \int_{-\infty}^{\infty}|x(t)|^2 dt = \int_{-\tau/2}^{\tau/2}\left(\frac{1}{2}\left(1+\cos\left(\frac{\pi t}{\tau}\right)\right)\right)^2 dt = \frac{\tau}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(1 + 2\cos(\pi u) + \cos^2(\pi u)\right)du$$

$$u = \frac{t}{\tau} \qquad du = \frac{dt}{\tau}$$

$$t = u\tau$$

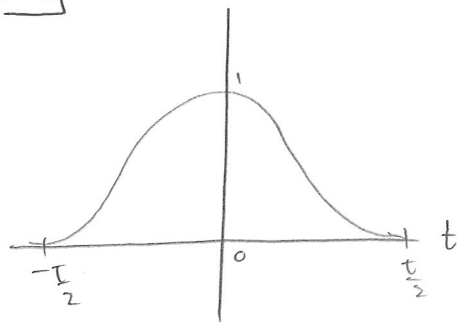
$$= \frac{\tau}{4}\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) + \frac{\tau}{4}\,\frac{2}{\pi}\left(\underbrace{\sin(\pi/2)}_{+1} - \underbrace{\sin\left(-\frac{\pi}{2}\right)}_{-1}\right) + \frac{\tau}{4}\int_{-\frac{1}{2}}^{1/2}\frac{1}{2}\left(1 + \cos(2\pi u)\right)du$$

$$= \frac{\tau}{4} + \frac{\tau}{\pi} + \frac{\tau}{4}\,\frac{1}{2}\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) + \frac{\tau}{4}\,\frac{1}{2}\,\frac{1}{2\pi}\left(\underbrace{\sin(\pi)}_{0} - \underbrace{\sin(-\pi)}_{0}\right)$$

$$= \tau\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{\pi}\right) = \boxed{\tau\left(\frac{3}{8} + \frac{1}{\pi}\right)}$$

Alternate version of problem 1:

1.a



1.b

$$x(t) = \frac{1}{2}\left(1 + \cos\left(\frac{2\pi t}{\tau}\right)\right)\Pi\left(\frac{t}{\tau}\right)$$

$$\Pi\left(\frac{t}{\tau}\right)e^{\pm 2\pi j t/\tau} \overset{\mathcal{F}}{\rightarrow} \tau\,\mathrm{sinc}\left(\tau\left(f \mp \frac{1}{\tau}\right)\right) = \tau\,\mathrm{sinc}(\tau f \mp 1)$$

$$\boxed{X(f) = \frac{\tau}{2}\,\mathrm{sinc}(\tau f) + \frac{\tau}{4}\,\mathrm{sinc}(\tau f - 1) + \frac{\tau}{4}\,\mathrm{sinc}(\tau f + 1)}$$

1.c

$$\int_{-\infty}^{\infty}|X(f)|^2 df = \int_{-\infty}^{\infty}|x(t)|^2 dt = \frac{1}{4}\int_{-\tau/2}^{t/2}\left|1 + \cos\left(\frac{2\pi t}{\tau}\right)\right|^2 dt = \frac{\tau}{4}\int_{-1/2}^{1/2}\left(1 + 2\cos(2\pi t) + \cos^2(2\pi t)\right)dt$$

$$= \frac{\tau}{4}\int_{-1/2}^{1/2}\left(1 + 2\cos(2\pi t) + \frac{1}{2} + \frac{1}{2}\cos(4\pi t)\right)dt$$

$$= \frac{t}{4}\left(\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) + \frac{2}{2\pi}\left(\cancel{\sin(\pi)}^{0} - \cancel{\sin(-\pi)}^{0}\right) + \frac{1}{2}\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) + \frac{1}{8\pi}\left(\cancel{\sin(2\pi)}^{0} - \cancel{\sin(-2\pi)}^{0}\right)\right)$$

$$= \frac{\tau}{4}\left(1 + \frac{1}{2}\right) = \boxed{\frac{3\tau}{8}}$$

1.b alternate continued

It can be shown that if a signal's n^{th} derivative is piecewise continuous, then its spectrum decays at least as fast as f^{-n} as $f \to \infty$.

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-2\pi j f t} dt$$

$$(2\pi j f)^n X(f) = \int_{-\infty}^{\infty} x^{(n)}(t) e^{-2\pi j f t} dt$$

$$|2\pi f|^n |X(f)| = \left| \int_{-\infty}^{\infty} x^{(n)}(t) e^{-2\pi j f t} dt \right| \leq \int_{-\infty}^{\infty} |x^{(n)}(t)| dt$$

$$|X(f)| \leq \frac{\int_{-\infty}^{\infty} |x^{(n)}(t)| dt}{|2\pi f|^n} = \frac{M}{|f^n|}$$ for some constant M if the integral exists

A stronger result is that if $x^{(n)}(t)$ is piecewise-constant, then $X(f)$ decays like $f^{-(n+1)}$. Clearly, signals like $\Pi(t)$ $(n=0)$, $\Lambda(t)$ $(n=1)$, and $\mathrm{sgn}(t)$ $(n=0)$ obey this rule. The raised cosine spectrum should decay like f^{-3}, because its 2^{nd} derivative is discontinuous. Does it?

$$X(f) = \frac{\tau}{4}\left(2\,\mathrm{sinc}(\tau f) + \mathrm{sinc}(\tau f + 1) + \mathrm{sinc}(\tau f - 1)\right)$$

$$\mathrm{sinc}(f+n) = \frac{\sin(\pi f + \pi n)}{\pi(f+n)} = \frac{\sin(\pi f)\cos(\pi n) + \cos(\pi f)\sin(\pi n)}{\pi(f+n)}$$

$\cos(\pi n) = (-1)^n$

$\sin(\pi n) = 0$

$$= \frac{\sin(\pi f)(-1)^n}{\pi(f+n)} = \frac{(-1)^n f}{f+n}\,\mathrm{sinc}(f)$$

$$X(f) = \frac{\tau}{4}\left(2 - \frac{\tau f}{\tau f - 1} - \frac{\tau f}{\tau f + 1}\right)\mathrm{sinc}(\tau f)$$

$$2 - \frac{q}{q-1} - \frac{q}{q+1} = \frac{2(q^2-1)}{q^2-1} - \frac{q^2+q}{q^2-1} - \frac{q^2-q}{q^2-1} = \frac{2q^2 - 2 - q^2 - q - q^2 + q}{q^2-1}$$

$$= \frac{2}{1-q^2}$$

$$X(f) = \frac{\tau}{2}\,\frac{\mathrm{sinc}(\tau f)}{1-(\tau f)^2} \leq \frac{1}{2\pi f(1-(\tau f)^2)} \sim \frac{1}{f^3}$$

So the raised cosine has a more localized spectrum than a rectangular pulse, which is usually desireable in communications.

2.

Signals of the form $ce^{2\pi j f t}$ (where c is a complex constant) are eigensignals of any LTI system, with $H(f)$ the eigenvalue.

So $y(t) = \sum_{k=-\infty}^{\infty} G[k]\, H(\frac{k}{T})\, e^{2\pi j k t/T}$

$\frac{k}{T} = 10^3 k$, so $H(\frac{k}{T}) \neq 0$ only when $k = \pm 3$.

$$y(t) = G[3]\, H(\tfrac{3}{T})\, e^{2\pi j 3t/T} + G[-3]\, H(-\tfrac{3}{T})\, e^{-2\pi j 3t/T}$$

$$= \frac{1}{\pi j 3} 2j\, e^{2\pi j 3t/T} + \left(\frac{-1}{\pi j 3}\right)(-2j)\, e^{-2\pi j 3t/T}$$

$$= \frac{2}{3\pi} 2\cos(6\pi t/T) = \boxed{\frac{4}{3\pi}\cos(6\pi t/T)}$$

3.

$x(t) = 2\,\text{sinc}(40t)$

$y(t) = 20\,\text{sinc}(40t - 200) = 20\,\text{sinc}(40(t-5)) = 10x(t-5)$

if $y(t) = 10x(t-5)$, then $Y(f) = 10e^{-2\pi j 5f} X(f)$

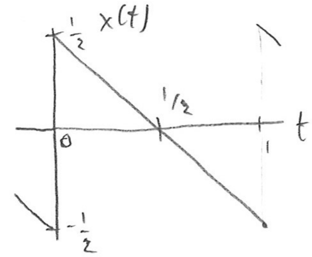
Also, $X(f) = \frac{2}{40}\Pi(\frac{f}{40})$, which is a box of width 40 (from -20 to 20).

We know that $H(f) = \frac{Y(f)}{X(f)} = 10e^{-2\pi j 5f}$ for $|f| < 20$. It is unknown outside that range, because the signal we sent in has no frequency content outside of $|f| < 20$.

$$H(f) = \begin{cases} 10e^{-2\pi j 5f} & |f| < 20 \\ \text{unknown} & \text{otherwise} \end{cases}$$

Probably the easiest way is straight integration with the analysis equation.

$$X[k] = \frac{1}{T}\int_0^T \left(\frac{1}{2} - t\right) e^{-2\pi j k t/T}\, dt \qquad T = 1$$



$$= \frac{1}{2}\, \frac{e^{-2\pi j k t}}{-2\pi j k}\Bigg]_{t=0}^{1} + -\int_0^1 t\, e^{-2\pi j k t}\, dt$$

$$\frac{1}{2}\, \frac{\overset{1}{\cancel{e^{-2\pi j k}}} - 1}{-2\pi j k} = 0 \qquad k \neq 0$$

$$-\int_0^1 t\, e^{-2\pi j k t}\, dt = -t\, \frac{e^{-2\pi j k t}}{-2\pi j k}\Bigg]_{t=0}^{t=1} + \int_0^1 \frac{e^{-2\pi j k t}}{-2\pi j k}\, dt$$

$u = t \qquad dv = e^{-2\pi j k t}\, dt$

$du = dt \qquad v = \dfrac{e^{-2\pi j k t}}{-2\pi j k}$

$$= \frac{-\overset{1}{\cancel{e^{-2\pi j k}}}}{-2\pi j k} + \frac{e^{-2\pi j k t}}{(-2\pi j k)^2}\Bigg]_{t=0}^{1} \qquad k \neq 0$$

$$= \frac{1}{2\pi j k} + \frac{1 - 1}{(-2\pi j k)^2} = \frac{1}{2\pi j k} \qquad k \neq 0$$

By inspection, $X[0] = \int_0^1 \left(\frac{1}{2} - t\right) dt = 0$

But let's be complete: $\frac{1}{2}t - \frac{1}{2}t^2\Big]_{t=0}^{1} = \left(\frac{1}{2} - \frac{1}{2}\right) - 0 = 0$

$$\boxed{X[k] = \begin{cases} \dfrac{1}{2\pi j k} & k \neq 0 \\ 0 & k = 0 \end{cases}}$$

5.1

$$x(t) = g(-3(t-2)) + g^{\dagger}(t) + \mathrm{Re}\left(g(t)e^{2\pi j 2t}\right)$$

$$g(t) \xrightarrow{\mathcal{F}} G(f)$$

$$g^{\dagger}(t) \xrightarrow{\mathcal{F}} G^{*}(-f)$$

$$g(-3t) \xrightarrow{\mathcal{F}} \frac{1}{3}G\left(-\frac{f}{3}\right)$$

$$g(-3(t-2)) \xrightarrow{\mathcal{F}} \frac{1}{3}G\left(-\frac{f}{3}\right)e^{-2\pi j 2f}$$

$$g(t)e^{2\pi j 2t} \xrightarrow{\mathcal{F}} G(f-2)$$

$$\mathrm{Re}\left(g(t)e^{2\pi j 2t}\right) = \frac{1}{2}g(t)e^{2\pi j 2t} + \frac{1}{2}\left(g(t)e^{2\pi j 2t}\right)^{*}$$

$$\xrightarrow{\mathcal{F}} \frac{1}{2}G(f-2) + \frac{1}{2}G^{\dagger}(f-2)$$

$$\boxed{X(f) = \frac{1}{3}G\left(-\frac{f}{3}\right)e^{-2\pi j 2f} + G^{*}(f) + \frac{1}{2}G(f-2) + \frac{1}{2}\left(G(f-2)\right)^{\dagger}}$$

Alternate way to do the last terms:

$$\mathrm{Re}\left(g(t)e^{2\pi j 2t}\right) = \frac{1}{2}g(t)e^{2\pi j 2t} + \frac{1}{2}g^{*}(t)e^{-2\pi j 2t}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{1}{2}G(f-2) + \frac{1}{2}G^{*}(-(f+2))$$

$$= \frac{1}{2}G^{*}(-f-2)$$

$$= \frac{1}{2}\left(G(f-2)\right)^{\dagger}$$

$$h(t) = g^{*}(t)$$

$$H(f) = G^{\dagger}(f) = G^{*}(-f)$$

$$g^{*}(t)e^{-2\pi j f_0 t} = h(t)e^{-2\pi j f_0 t} \Rightarrow H(f+f_0) = G^{*}(-(f+f_0)) = G^{*}(-f-f_0) = \left(G(f-f_0)\right)^{\dagger}$$